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On Proximinality and Sets of Operators. I. Best Approximation by Finite Rank Operators

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In this paper it is shown that $K_n(X, C_0, (Q))$ is proximinal in $L(X, C_0(Q))$ when X^* is uniformly convex, thus solving Problem 5.2.3 of Deutsch, Mach and, Saatkamp (*J. Approx. Theory* **33** (1981), 199–213). The solutions of Problem 5.2.5 of Deutsch, *et al.* and the of problem 5.B of Franchetti and Cheney (*Boll. Un Mat. Ital. B*(*5*) **18** (1981), 1003–1015) are also included.

INTRODUCTION

If X and Y are normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y, $K(X, Y)$ the set of all compact operators in $L(X, Y)$ and $K_n(X, Y)$ the set of all operators of rank $\leqslant n$ in $L(X, Y)$.

If A is a closed subset of the normed linear space X, then A is said to be the proximal in X if, for each $x \in X$, there is $y_0 \in A$ such that

$$\|x - y_0\| = d(x, A) = \inf\{\|x - y\|;\ y \in A\}.$$

If C is proximal in X the set-valued function

$$P_C\colon\quad X \to 2^C \text{ defined by}$$

$$P_C(x) = \{y \in C;\ \|x - y\| = d(x, C)\}$$

is called the metric projection from X onto C. If there is a continuous function $g\colon X \to C$, such that $g(x) \in P_C(x)$ for each $x \in X$, then g is called a continuous selection for the metric projection P_C.

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The proximinality of $K(X, Y)$ in $L(X, Y)$ has been studied by several authors, but only a few authors have worked on the proximinality of $K_n(X, Y)$ in $K(X, Y)$ or $L(X, Y)$.

It is easy to show that if Y^* is the dual space of Y, then for any normed linear space X, the set $K_n(X, Y^*)$ is proximinal in $L(X, Y^*)$. Fakhoury [5] used the fact that for $1 < p < \infty$, the metric projection from l_p onto any of its finite dimensional subspaces is ω^*-continuous, to prove that for any Hausdorff topological space Q, the set $K_n(l_p, C_0(Q))$ is proximinal in $L(l_p, C_0(Q))$. Deutsch *et al.* [3] used the fact that the metric projection from any strictly convex space X onto any of its finite-dimensional subspaces is continuous, to prove that if X^* is strictly convex and if Q is a Hausdorff topological space, then the set $K_n(X, C_0(Q))$ is a proximinal in $K(K, C_0(Q))$. They also proved that if X^* is uniformly convex then $K_n(X, c_0)$ is proximinal in $L(X, c_0)$.

This paper contains a further study for the proximinality of $K_n(X, Y)$ in $L(X, Y)$ and $K(X, Y)$. The paper is divided into three sections:

Section one contains the necessary known results that will be used later, it contains also some definitions and notations. In section two it is shown that if X^* is uniformly convex and Q is a locally compact Hausdorff space, then $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$. This result includes the results of Fakhoury [5] and Deutsch *et al.* [3], and gives a positive solution to Problem 5.2.3 of Deutsch *et al.* [3]. Section three deals with the proximinality of $K_n(X, C_0(Q))$ in $K(X, C_0(Q))$ when $\dim X < \infty$. The main result is that, if X is a finite dimensional Banach space, then $K_n(X, C_0(Q))$ is proximinal in $K(K, C_0(Q))$ for each locally compact Hausdorff space Q, if and only if for each n-dimensional subspace Y of X^*, the metric projection P_y from X^* onto Y has a continuous selection. It is shown also that there is a Banach space X, a compact Hausdorff space Q and two positive integers $m \neq n$ such that $K_m(X, C(Q))$ is proximinal in $L(X, C(Q))$ whereas $K_n(X, C(Q))$ is not. This gives a negative solution to Problem 5.2.5 of Deutsch *et al.* [3]. The space $C(Q, X)$ of all continuous bounded functions from Q to X, appears naturally in the proof of those results, and it is shown that there is a finite dimensional Banach space X, a subspace Y of X and a compact Hausdorff space Q, such that the set $C(Q, Y)$ is not proximal in $C(Q, X)$. This gives a negative solution to the problem 5.B of Franchetti and Cheney [6].

1. Definitions and Known Results

If C is a subset of the normed linear space X, and $x \in X$ then $d(x, C) = \inf\{\|x - y\|; y \in C\}$ is the distance of x from C. If there is y_0 such that $\|x - y_0\| = d(x, C)$ then y_0 is called a "best approximation" from C. C is said to be proximal in X if for each $x \in X$ there is $y \in C$ such that $\|x-y\| = d(x, C)$. It is easy to show that if C is a compact subset of the Banach space X, then C is proximinal in X, it follows from this, that every finite-dimensional subspace Y of X is proximinal in X.

If A and C are two subsets of the normed linear space X then

$$\delta(A, C) = \sup\{d(x, C); x \in A\}$$

is the deviation of A from C. For each nonnegative integer $n \geqslant 0$.

$$d_n(A, X) = \inf\{\delta(A, N); N \text{ is an } n\text{-dimensional subspace of } X\}$$

is the Kolmogrov n-width of A in X. If there is a subspace N_0 of X of dimension n, such that $d_n(A, X) = \delta(A, N_0)$, then N_0 is called an "extremal subspace for $d_n(A, X)$," and we say that $d_n(A, X)$ is attained.

If Q is a Hausdorff topological space and X is a normed linear space, then $B(Q, X)$ denotes the set of all bounded functions from Q to X. If τ is a topology on X then $C(Q, (X, \tau))$ denotes the set of all bounded functions from Q to X which are continuous with respect to τ. Furthermore, $C_0(Q.X) = \{f \in C(Q, (X, \|\cdot\|)); \forall \varepsilon > 0$, the set $\{q \in Q; \|f(q)\| \geqslant \varepsilon\}$ is compact. When $X = R$ "the set of real numbers" then $B(Q, R)$ is denoted by $B(Q)$, and $C_0(Q, R)$ is denoted by $C_0(Q)$. If X^* is the dual space of X then

$$C_0(Q, (X^*, \omega^*)) = \{f \in C(Q, (X^*, \omega^*)); \hat{x} \circ f \in C_0(Q)\ \forall x \in X\},$$

where $\hat{x}$ is the image of x under the canonical injection of X in X^{**}.

The following theorem will be used frequently in the next two sections.

1.1. THEOREM. *Let Q be a locally compact Hausdorff space and let X be a normed linear space. There is a mapping $\alpha\colon L(X, C_0(Q)) \to C_0(Q, (X^*, \omega^*))$ defined by*

$$\alpha(T)(q)(x) = T(x)(q) \qquad \text{for} \quad T \in L(X, C_0(Q)), \qquad q \in Q \text{ and } x \in X.$$

The mapping α is an isometric isomorphism from $L(X, C_0(Q))$ onto $C_0(Q, (X^*, \omega^*))$. Furthermore, $\alpha(K(X, C_0(Q))) = C_0(Q, X^*)$ and

$$\alpha(K_n(X, C_0(Q))) = \bigcup_N \{C_0(Q, N); N \text{ is an } n\text{-dimensional subspace of } X^*\}.$$

The proof of this theorem, when Q is a compact Hausdorff space, can be found in Dunford and Schwartz [4, p.490].

From Theorem 1.1 one can obtain the following lemma:

1.2. LEMMA. *Let Q be a locally compact Hausdorff space and let X be a normed linear space.*

(a) *$K(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$, iff $C_0(Q, X^*)$ is proximinal in $C_0(Q, (X^*, \omega^*))$.*

(b) *$K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$], iff the set*

$$\bigcup_N \{C_0(Q, N);\ N \text{ is an } n\text{-dimensional subspace of } X^*\}$$

is proximinal in $C_0(Q, (X^*, \omega^*))$ [resp. $C_0(Q, X^*)$]

1.3. Definition. Let Q be a locally compact Hausdorff space and let X be a normed linear space. If $f \in B(Q, X)$ and n is a nonnegative integer let $a_n(f) = \inf\{d(f, C_0(Q, N));\ N$ is an n-dimensional subspace of $X\}$.

It follows from Lemma 1.2 that $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$] iff $a_n(f)$ is attained for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $C_0(Q, X^*)$]. The proximinality of $K_n(X, C_0(Q)$ in $K(X, C_0(Q))$ was studied by Deutsch, Mack and Saatkamp [3], and the following theorem is due to them.

1.4. Theorem. *Let X be a normed linear space, Q a locally compact Hausdorff space and let $n \geqslant 1$ be a positive integer. If the metric projection P_Y from X^* onto any n-dimensional subspace Y of X^* has a continuous selection, then the set $K_n(X, C_0(Q))$ is proximinal in $K(X, C_0(Q))$.*

Theorem 1.4 is not vacuous since Brown [1] showed that, if X is strictly convex space, or if X is a polyhedral finite-dimensional Banach space, then the metric projection from X onto any finite-dimensional subspace Y of X has a continuous selection.

The closed subspace M of $C_0(Q, X)$ is called a closed $C_0(Q)$-submodule, if for each $f \in M$, and each $g \in C_0(Q)$, the element $g \cdot f \in M$, where $g \cdot f(q) = g(q) \cdot f(q)$. It is obvious that if N is a closed subspace of X, then $C_0(Q, N)$ is a closed $C_0(Q)$-submodule in $C_0(Q, X)$. The following theorem deals with the proximinality of $C_0(Q)$-submodules in $B(Q, X)$.

1.5. Theorem. (Lau [7]). *If Q is a locally compact Hausdorff space, and X is a uniformly convex space, then for any bounded function $f: Q \to X$ and any closed $C_0(Q)$-submodule M in $C_0(Q, X)$, there is $g_0 \in M$ such that*

$$\|f - g_0\| = d(f, M).$$

2. $K_n(X, C_0(Q))$ IS PROXIMAL IN $L(X, C_0(Q))$ WHEN X^* IS UNIFORMLY CONVEX

In this section it will be shown that if X^* is uniformly convex and Q is a locally compact Hausdorff space, then $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$. After appealing to Lemma 1.2, the main step in the proof is to find for each $f \in C_0(Q, (X^*, \omega^*))$ an n-dimensional subspace N_0 of X^*, such that

$$d(f, C_0(Q, N_0)) = a_n(f).$$

Lemma 2.4 gives an estimate for $d(f, C_0(Q, N))$, and Lemma 2.8 shows that for some n-dimensional subspace N_0 of X^* this estimate is less than or equal to $a_n(f)$.

2.1. DEFINITION. Let X be a Banach space, Q a Hausdorff topological space and $f: Q \to X$ a bounded function. For $q_0 \in Q$, $x \in X$ and $A \subseteq X$, let

(1) $r(f, q_0, x) = \inf_U \sup_{q \in U} \|f(q) - x\|$, where U ranges through all the neighbourhoods of q_0 in Q,

(2) $r(f, q_0, A) = \inf_{x \in A} r(f, q_0, x)$,

(3) $r(f, A) = \sup_{q_0 \in Q} r(f, q_0, A)$.

2.2. DEFINITION. For Q a locally compact Hausdorff space, X a Banach space and $f: Q \to X$ a bounded function define

$$b(f) = \inf\{\alpha \geqslant 0;\ \text{there is a compact subset } K \text{ in } Q \text{ with } \|f(q)\| < \alpha \text{ for all } q \notin K\}.$$

2.3. LEMMA. *Let X be a Banach space, Q a locally compact Hausdorff space and $f: Q \to X$ a bounded function. For any positive integer $n \geqslant 1$, any n-dimensional subspace N of X and any $q \in Q$.*

(1) $r(f, q, X) \leqslant r(f, q, N) \leqslant r(f, N) \leqslant d(f, C_0(Q, N))$,

(2) $b(f) \leqslant d(f, C_0(Q, N))$,

(3) $r(f, q, X) \leqslant a_n(f)$.

Proof. (1) It is obvious that $r(f, q, X) \leqslant r(f, q, N) \leqslant r(f, N)$. So it is enough to show that $r(f, N) \leqslant d(f, C_0(Q, N))$. Let $\varepsilon > 0$ be given. There is $g \in C_0(Q, N)$ such that

$$\|f - g\| \leqslant d(f, C_0(Q, N)) + \frac{\varepsilon}{2}.$$

Let $q_0 \in Q$. There is a neighbourhood U_0 of q_0 such that

$$\|g(q) - g(q_0)\| \leqslant \frac{\varepsilon}{2} \qquad \text{for all} \quad q \in U_0,$$

thus

$$r(f, q_0, N) \leqslant r(f, q_0, g(q_0)) \leqslant \sup_{q \in U_0} \|f(q) - g(q_0)\|$$

$$\leqslant \sup_{q \in U_0} \|f(q) - g(q)\| + \frac{\varepsilon}{2} \leqslant \|f - g\| + \frac{\varepsilon}{2} \leqslant d(f, C_0(Q, N)) + \varepsilon.$$

Since q_0 and ε are arbitrary, it follows that $r(f, N) \leqslant d(f, C_0(Q, N))$.

(2) Let $\varepsilon > 0$ be given. There is $g \in C_0(Q, N)$ such that

$$\|f - g\| \leqslant d(f, C_0(Q, N)) + \frac{\varepsilon}{2}.$$

For the same ε, there is a compact subset K of Q with $\|g(q)\| < \varepsilon/2$ for each $q \notin K$. Thus

$$b(f) \leqslant \sup_{q \notin K} \|f(q)\| \leqslant \sup_{q \notin k} \|f(q) - g(q)\| + \frac{\varepsilon}{2} \leqslant \|f - g\| + \frac{\varepsilon}{2}$$

$$\leqslant d(f, C_0(Q)) + \varepsilon.$$

(3) Follows from (1). ∎

2.4. LEMMA. *Let X be a Banach space, Q a locally compact Hausdorff space and $f\colon Q \to X$ a bounded function. If N is an n-dimensional subspace of X then*

$$d(f, C_0(Q, N)) = \max\{r(f, N), b(f)\}.$$

Proof. By Lemma 2.3 $d(f, C_0(Q, N)) \geqslant \max\{r(f, N), b(f)\}$, thus it is enough to show that $\max\{r(f, N), b(f)\} \geqslant d(f, C_0(Q, N))$. Let $\varepsilon > 0$ be given. There is a compact set $K \subseteq Q$ such that $\|f(q)\| \leqslant b(f) + \varepsilon$ for each $q \notin K$. By the compactness of K and the definition of $r(f, q, N)$, one can find $\{q_i\}_{i=1}^m \subseteq K$, $\{y_i\}_{i=1}^m \subseteq N$ ans $\{U_i\}_{i=1}^m$ an open cover for K in Q, such that:

(1) $q_i \in U_i$ for each $i = 1, 2, ..., m$,

(2) $\|f(q) - y_i\| \leqslant r(f, q_i, N) + \varepsilon$ for each $q \in U_i$, and

(3) U_i is relatively compact for for each $i = 1, 2, ..., m$.

Let $U_{m+1} = Q/K$ and let $y_{m+1} = 0$, then for each $q \in U_{m+1}$

$$\|f(q) - y_{m+1}\| = \|f(q)\| \leqslant b(f) + \varepsilon.$$

Let $\{\phi_i\}_{i=1}^{m+1}$ be a partition of unity corresponding to $\{U_i\}_{i=1}^{m+1}$, and define the function $g\colon Q \to N$ by

$$g(q) = \sum_{i=1}^{m+1} \phi_i(q)\, y_i \qquad \text{for} \quad q \in Q.$$

Then g is continuous, and since $\bigcup_{i=1}^{m} U_i$ is relatively compact, it follows that $g \in C_0(Q, N)$. It follows from the above argument that, for each $q \in Q$, if $q \in U_i$ then $\|f(q) - y_i\| \leqslant \max\{r(f, N), b(f)\} + \varepsilon$. Also if $\phi_i(q) \neq 0$ then $q \in U_i$. Thus for each $q \in Q$

$$\begin{aligned} \|f(q) - g(q)\| &= \left\| \sum_{i=1}^{m-1} \phi_i(q)(f(q) - y_i) \right\| \\ &= \left\| \sum \phi_i(q)(f(q) - y_i) \right\| \qquad \{i; \phi_i(q) \neq 0\} \\ &\leqslant \sum \phi_i(q) \|f(q) - y_i\| \qquad \{i; \phi_i(q) \neq 0\} \\ &\leqslant \max\{r(f, N), b(f)\} + \varepsilon \end{aligned}$$

So $\|f - g\| \leqslant \max\{r(f, N), b(f)\} + \varepsilon$. Since ε is arbitrary then

$$d(f, C_0(Q, N)) \leqslant \max\{r(f, N), b(f)\}. \quad \blacksquare$$

2.5. Corollary. *Let X be a Banach space, Q a locally compact Hausdorff space and let N be an n-dimensional subspace of X. For each $f \in C_0(Q, X)$*

$$d(f, C_0(Q, N)) = \delta(f(Q), N).$$

Proof. If $f \varepsilon C_0(Q, X)$ then $b(f) = 0$, so by Lemma 2.4

$$d(f, C_0(Q, N)) = r(f, N).$$

Thus to complete the proof it is enough to show that for each $q \in Q$ the equality $d(f(q), N) = r(f, q, N)$ holds, but this follows from the continuity of f.

2.6. Definition. Let X be a Banach space, Q a Hausdorff topological space and $f\colon Q \to X$ a bounded function. For each $q \in Q$ define

$$\Gamma_n(f, q) = \{x \in X; r(f, q, x) \leqslant a_n(f)\},$$

and for $i \geqslant 1$ define

$$\Gamma_n(f, q, i) = \left\{x \in X; r(f, q, x) \leqslant a_n(f) + \frac{1}{i}\right\}.$$

It is clear that if there is an n-dimensional subspace N of X, such that $C_0(Q, N)$ contains a function g for which $\|f - g\| = a_n(f)$, then

$$g(q) \in \Gamma_n(f, q) \cap N,$$

that is, for each $q \in Q$ the set $\Gamma_n(f, q) \cap N$ is not empty.

It will be shown in Lemma 2.8 that if X is a reflexive Banach space, then there is an n-dimensional subspace N_0 such that $\Gamma_n(f, q) \cap N_0 \neq \varnothing$ for each $q \in Q$, this and the results of Lau [7] will be used in Theorem 2.10 to show that, if X is uniformly convex then $a_n(f)$ is attained.

2.7. Lemma. *Let X be a Banach space, Q a Hausdorff topological space and $f: Q \to X$ a bounded function. For each $q \in Q$, the sets $\Gamma_n(f, q)$ and $\Gamma_n(f, q, i)$ are closed and convex. If X is reflexive then they are ω-compact. Furthermore, for each $q \in Q$ the following statements are satisfied*:

(1) *If $i < j$ then $\Gamma_n(f, q, j) \subseteq \Gamma_n(f, q, i)$;*

(2) $\bigcap_{i=1}^{\infty} \Gamma_n(f, q, i) = \Gamma_n(f, q)$;

(3) *if X is reflexive, $\{x_i\}_{i=1}^{\infty}$ is a bounded sequence in X such that $x_i \in \Gamma_n(f, q, i)$ for each $i \geqslant 1$, and x_0 is a ω-cluster point for the sequence $\{x_i\}_{i=1}^{\infty}$, then $x_0 \in \Gamma_n(f, q)$.*

Proof. (2) It is clear that, for each positive integer $i \geqslant 1$, $\Gamma_n(f, q) \subseteq \Gamma_n(f, q, i)$. Thus $\Gamma_n(f, q) \subseteq \bigcap_{i=1}^{\infty} \Gamma_n(f, q, i)$. On the other hand if $x \in \bigcap_{i=1}^{\infty} \Gamma_n(f, q, i)$, then $r(f, q, x) \leqslant a_n(f) + (1/i)$ for each $i \geqslant 1$ and, therefore $r(f, q, x) \leqslant a_n(f)$. Thus $x \in \Gamma_n(f, q)$.

(3) Since X is reflexive the sets $\Gamma_n(f, q, i)$ and $\Gamma_n(f, q)$ are ω-compact. Assume that x_0 is a ω-cluster point for the sequence $\{x_i\}_{i=1}^{\infty}$ and that $x_i \in \Gamma_n(f, q, i)$ for each $i \geqslant 1$, it follows from part (1) of this lemma that for each positive integer $j_0 \geqslant 1$, the subsequence $\{x_i\}_{i=j_0}^{\infty}$ lies in $\Gamma_n(f, q, j_0)$, but $\Gamma_n(f, q, j)$ is ω-compact, so $x_0 \in \Gamma_n(f, q, j_0)$, therefore,

$$x_0 \in \bigcap_{j_0=1}^{\infty} \Gamma_n(f, q, j_0) = \Gamma_n(f, q).$$

The proof of the next lemma depends on the existence of an Auerbach basis in each finite-dimensional Banach space. Let X be a Banach space of dimension n. An Auerbach system on X is a basis $\{x_i\}_{i=1}^{n}$ in X with $\|x_i\| = 1$ for each i, and a basis $\{x_i^*\}_{i=1}^{n}$ in X^* with $\|x_i^*\| = 1$ for each i, such that $x_i^*(x_j) = \delta_{ij}$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. In this case $\{x_i\}_{i=1}^n$ is called an Auerbach basis in X. ▌

2.8. Lemma. *Let X be a reflexive Banach space, Q a Hausdorff space and $f: Q \to X$ a bounded function. For each nonnegative integer $n \geqslant 0$ there is an n-dimensional subspace N_0 of X, such that for each $q \in Q$ the set $\Gamma_n(f, q) \cap N_0$ is not empty.*

Proof. For each positive integer $i \geqslant 1$, there is an n-dimensional subspace N_i and a function $g_i \in C_0(Q, N_i)$ such that $\|f - g_i\| \leqslant a_n(f) + (1|i)$. Let $\{x_i^1, x_i^2, \ldots, x_i^n\}$ an Auerbach basis for N_i. There is $\{g_i^1, \ldots, g_i^n\} \subseteq C_0(Q)$ such that $g_i(q) = \sum_{k=1}^n g_i^k(q)\, x_i^k$. Clearly for each $q \in Q$ the point $g_i(q) = \sum_{k=1}^n g_i^k(q)\, x_i^k \in \Gamma_n(f, q, i)$. Since $\{x_i^1, \ldots, x_i^n\}$ is an Auerbach basis, the sequence $\{(g_i^1, \ldots, g_i^n)\}_{i=1}^\infty$ is a bounded sequence in $\prod_{i=1} B(Q)$. Let $Z = \prod_{i=1}^n X \times \prod_{i=1}^n B(Q)$ be the topological vector space obtained by giving X its weak topology, and identifying $B(Q)$ in the standard way with $l_1^*(Q)$ with the ω^*-topology. Then the sequence $\{(x_i^1, \ldots, x_i^n, g_i^1, \ldots, g_i^n)\}_{i=1}^\infty$ has a cluster point $(x_0^1, \ldots, x_0^n, g_0^1, \ldots, g_0^n)$ in Z. Let N_0 be an n-dimensional subspace of X that contains $\{x_0^1, \ldots, x_0^n\}$. It will be shown that for any $q_0 \in Q$ the point $\sum_{k=1}^n g_0^k(q_0)\, x_0^k \in \Gamma_n(f, q_0) \cap N_0$. To prove this it is enough by Lemma 2.7 to show that the point $\sum_{k=1}^n g_0(q_0)\, x_0^k$ is a ω-cluster point for the sequence $\{\sum_{k=1}^n g_i^k(q_0)\, x_i^k\}_{i=1}^\infty$ in X. Let $y \neq 0$ be an element in X^*, and let $\varepsilon > 0$ be given. Since $(x_0^1, \ldots, x_0^n, g_0^1, \ldots, g_0^n)$ is a cluster point for the sequence $\{(x_i^1, \ldots, x_i^n, g_i^1, \ldots, g_i^n)\}_{i=1}^\infty$, it follows that there is an infinite subset M of positive integers, such that for each $i \in M$

(1) $|\sum_{k=1}^n g_0^k(q_0)\, y(x_0^k) - \sum_{k=1}^n g_0^k(q_0)\, y(x_i^k)| \leqslant \varepsilon/2$, and

(2) for each $k = 1, 2, \ldots, n$,

$$|g_0^k(q_0) - g_i^k(q_0)| \leqslant \frac{\varepsilon}{2n.\|y\|}.$$

Then for $i \in M$

$$\begin{aligned}
&\left|\sum_{k=1}^n g_0^k(q_0)\, y(x_0^k) - \sum_{k=1}^n g_i^k(q_0)\, y(x_i^k)\right| \\
&\quad \leqslant \left|\sum_{k=1}^n g_0^k(q_0)\, y(x_0^k) - \sum_{k=1}^n g_0^k(q_0)\, y(x_i^k)\right| \\
&\quad\quad + \left|\sum_{k=1}^n g_0^k(q_0)\, y(x_i^k) - \sum_{k=1}^n g_i^k(q_0)\, y(x_i^k)\right| \\
&\quad \leqslant \tfrac{\varepsilon}{2} + \frac{n\|y\| \cdot \varepsilon}{2n\|y\|} = \varepsilon.
\end{aligned}$$

Thus the point $\sum_{k=1}^{n} g_0(q_0)\, x_0^k$ is a ω-cluster point for the sequence $\{\sum_{k=1}^{n} g_i^k(q_0)\, x_i^k\}_{i=1}^{\infty}$ in X. ∎

2.9. COROLLARY. *Let X, Q, and f be as in Lemma 2.8. There is an n-dimensional subspace N_0 of X, such that*

$$r(f, N_0) \leqslant a_n(f).$$

Proof. By Lemma 2.8 there is an n-dimensional subspace N_0 of X such that $N_0 \cap \Gamma_n(f, q) \neq \emptyset$ for each $q \in Q$. Thus for each $q \in Q$ there is $y \in N_0$ satisfying the inequality; $r(f, q, y) \leqslant a_n(f)$, but then

$$r(f, N_0) = \sup_{q \in Q} r(f, q, N_0) \leqslant a_n(f). \quad ∎$$

2.10. THEOREM. *If X is a uniformly convex space, Q is a locally compact Hausdorff space and f: $Q \to X$ is a bounded function then $a_n(f)$ is attained.*

Proof. By Lemma 2.4 for any n-dimensional subspace N of X

$$d(f, C_0(Q, N)) = \max\{r(f, N), b(f)\},$$

and by Corollary 2.9 there is an n-dimensional subspace N_0 of X satisfying $r(f, N_0) \leqslant a_n(f)$. Thus by Lemma 2.3

$$\begin{aligned} a_n(f) \leqslant d(f, C_0(Q, N_0)) &= \max\{r(f, N_0), b(f)\} \\ &\leqslant \max\{a_n(f), b(f)\} = a_n(f). \end{aligned}$$

So $a_n(f) = d(f, C_0(Q, N_0))$, hence by Theorem 1.5 and the fact that $C_0(Q, N_0)$ is a closed submodule in $C_0(Q, X)$, it follows that there is $g_0 \in C_0(Q, N_0)$ such that $\|f - g\| = d(f, C_0(Q, N_0)) = a_n(f)$. ∎

2.11. COROLLARY. *If X^* is uniformly convex space and Q is a locally compact Hausdorff space then for each nonnegative integer $n \geqslant 0$ the set $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$.*

Proof. A consequence of Theorem 2.10 and Lemma 1.2. ∎

Corollary 2.11 was known before for $X^* = l_p$ $1 < p < \infty$ and for $C_0(Q) = c_0$. Fakhoury [5] proved that for $1 < p < \infty$ the set $K_n(l_p, C_0(Q))$ is proximinal in $L(l_p, C_0(Q))$, and Deutsch, Mach and Saatkamp [3] proved that $K_n(X, c_0)$ is proximinal in $L(X, c_0)$ when X^* is uniformly convex. This corollary gives a positive solution to the problem 5.2.3 in Deutsch Mach and Saatkamp [3].

3. The Proximinality of $K_n(X, C_0(Q))$ in $K(X, C_0(Q))$ when X Is of Finite Dimension

3.1. Lemma. *Let X be a finite dimensional Banach space, M and N be two subspaces of X, such that $M+N=X$ and $M \cap N=\{0\}$, let S_M be the unit sphere of M and let P_N be the metric projection from X onto N. Then any continuous selection for $P_{N|S_M}$ can be extended to a continuous selection for P_N.*

Proof. Assume that $g: S_M \to N$ is a continuous selection for $P_{N|S_M}$ and define $f: M \to N$ by

$$f(x)=\begin{cases} \|x\| \, g\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

The f is a continuous selection for $P_{N|M}$. Since $\dim X<\infty$, $M+N=X$ and $M \cap N=\{0\}$, it follows that there is bounded projection P: $X \to N$ with $P(M)=\{0\}$ Let $h=f \circ (I-P)+P$. The h is a continuous function from X onto N, and for each $x \in M$ and each $y \in N$

$$h(x+y)=f(x)+y \in P_N(x)+y=P_N(x+y)$$

So h is a continuous selection for the metric projection P_N. ∎

3.2. Lemma. *Let X be a finite-dimensional Banach space, and let N be a subspace of X. If the metric projection P_N has no continuous selection, then there is a compact Hausdorff space Q', such that $C(Q', N)$ is not proximinal in $C(Q', X)$, that there is a function $f \in C(Q', X)$, such that $d(f, C(Q', N))$ is not attained.*

Proof. Let M be a subspace of X such that $M+N=X$ and $M \cap N=\{0\}$, and let $Q'=S_M$. By Lemma 3.1 there is no continuous function $g: Q' \to N$, such that $g(x) \in P_N(x)$ for each $x \in Q'$. Since $M \cap N=\{0\}$ then, for each $x \in Q'$, $d(x, N) \neq 0$. Define $f: Q' \to X$ by

$$f(x)=\frac{x}{d(x, N)}.$$

Since $d(x, N)$ is a continuous function, it follows that the function f is continuous, and $\delta(f(Q'), N)=1$. Therefore, by Corollary 2.5

$$d(f, C(Q', N))=1.$$

Assume that there is a continuous function $h: Q' \to N$, such that $\|f-h\| = 1$, and define $g: Q' \to N$ by

$$g(x) = d(x, N) \cdot h(x).$$

then g is continuous and for each $x \in Q'$

$$\begin{aligned}\|x - g(x)\| &= \|d(x, N)\, f(x) - d(x, N)\, h(x)\| \\ &= d(x, N)\|f(x) - h(x)\| \leqslant d(x, N).\end{aligned}$$

But this contradicts Lemma 3.1, so $C(Q', N)$ is not proximinal in $C(Q', X)$. ∎

3.3. LEMMA. *Let X be a finite-dimensional Banach space, let $n \geqslant 1$ be a positive integer such that* $\dim X > n$, *and let N be an n-dimensional subspace of X. There is a compact subset A of X, such that $d_n(A, X) \neq 0$ and N is the unique extremal subspace for $d_n(A, X)$.*

Proof. Let B_X [resp. B_N] be the closed unit ball of X [resp. N], and let A be the balanced convex hull of $B_X + B_N$, then $d_n(A, X) = \delta(A, N) = 1$. It will be shown that N is the unique extremal subspace for $d_n(A. X)$. Let $N' \neq N$ any n-dimensional subspace of X, then $\delta(B_N, N') > 0$. Let $y \in B_N$ be such that $d(y, N') > 0$, let $y' \in N'$ be such that $\|y - y'\| = d(y, N')$, and let $x = y + ((y - y')/\|y - y'\|)$. Then $x \in A$, and

$$d(x, N') = d\left(y - y' + \frac{(y - y')}{\|y - y'\|} + y', N'\right) = 1 + d(y, N') > 1.$$

Thus $\delta(A, N') \geqslant d(x, N') > 1$, therefore N is the unique extremal subspace for $d_n(A, X)$. ∎

3.4. LEMMA. *Let X be a finite-dimensional Banach space, and let $n \geqslant 1$ be a positive integer. If there is an n-dimensional subspace N of X^*, such that the metric projection P_N has no continuous selection, then there is a compact Hausdorff space Q, such that $K_n(X, C(Q))$ is not proximinal in $L(X, C(Q))$.*

Proof. Since X is of finite dimension then by Lemma 1.2 it is enough to show that there is a compact Hausdorff space Q, such that the set

$$\{g \in C(Q, X^*);\ g(Q) \text{ lies in an } n\text{-dimensional subspace of } X^*\}$$

is not proximinal in $C(Q, X^*)$.

Let Q' and f be as in Lemma 3.2 applied to X^* and N, let A be as in Lemma 3.3, and let Q be the disjoint topological union of Q' and A.

Without loss of generality assume that $d(f, C(Q', N)) = 1$ and $d_n(A, X) = 1$. Define $h: Q \to X$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in Q' \\ x & \text{if } x \in A. \end{cases}$$

Then h is continuous, and

$$a_n(h) \leqslant d(h, C(Q, N)) = \delta(h(Q), N) = 1.$$

Let N' be an n-dimensional subspace of X^* such that $d(h, C(Q, N')) \leqslant 1$, and assume that there is $g \in C(Q, N')$ such that $\|h - g\| \leqslant 1$, then

$$\delta(A, N') \leqslant \|h|_A - g|_A\| \leqslant 1.$$

so by Lemma 3.3 $N' = N$, but then $g|_{Q'}$ is continuous on Q' and

$$\|f - g|_{Q'}\| \leqslant d(f, C(Q', N)),$$

which contradicts Lemma 3.2. So $K_n(X, C(Q))$ is not proximinal in $L(X, C(Q))$. ∎

3.5. THEOREM. *Let X be a finite-dimensional Banach space. Then $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$ for each locally compact Hausdorff space Q, iff the metric projection from X^* onto any of its n-dimensional subspaces has a continuous selection.*

Proof. Since $\dim X < \infty$ then $L(X, C_0(Q)) = K(X, C_0(Q))$, therefore, the theorem follows from Theorem 1.4 and Lemma 3.4. ∎

3.6. *Note.* Brown [2] proved that there is a finite-dimensional space, that contains subspaces for which the metric projection has no continuous selection. This means that Lemma 3.2, Lemma 3.4, and Theorem 3.5 are not vacuous.

3.7. COROLLARY. *Let X be an m-dimensional Banach space. For any locally compact Hausdorff space Q, $K_{m-1}(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$.*

Proof. Follows from Theorem 3.5 and the fact that if N is a hyperplane in a finite-dimensional Banach space X then the metric projection P_N from X onto N has a continuous selection. ∎

3.8. COROLLARY. *There is a finite-dimensional Banach space X, a subspace N of X and a compact Hausdorff space Q, such that $C(Q, N)$ is not proximinal in $C(Q, X)$.*

Proof. Follows from Lemma 3.2 and Note 3.6. ∎

3.9. COROLLARY. *There is a finite dimensional Banach space X, a compact Hausdorff space Q and two positive integers m and n, such that $K_n(X, C(Q))$ is not proximinal in $L(X, C(Q))$, whereas $K_m(X, C(Q))$ is proximinal in $L(X, C(Q))$.*

Proof. Follows from Theorem 3.5, Note 3.6, and Corollary 3.7. ∎

Corollary 3.8 gives a negative solution to problem 5.B of Franchetti and Cheney [6], and corollary 3.9 gives a solution to Problem 5.2.5. of Deutsch *et al.* [3].

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